Bi-State Math Colloquium

Who: Dave Boyles
Where: Thursday, April 11, 4:00 pm
When: Ottensman 126, UW-Platteville

The Bott-Milnor-Kervaire 1,2,4,8 theorem

A real division algebra is a finite-dimensional vector space over the real numbers with a well-behaved multiplication of vectors. The prototype is the complex number field, two-thousand years in the making, with fundamental contributions by Heron of Alexandria, Cardan, Bombelli, Descartes, Leibniz, Euler, Argand, Wessel, Gauss, et. al.

Hamilton - inventor of Hamiltonian mechanics, which opened the door for quantum mechanics - knew by the early 1830s that the complex number field could be interpreted as a vector space of pairs of real numbers, with a specific way of multiplying them:

$$(a, b) \times (c, d) = (ac - bd, ad + bc)$$

and a division formula:

$$(a, b)/(c, d) = \left(\frac{ac + bd}{c^2 + d^2}, \frac{-ad + bc}{c^2 + d^2}\right)$$

He then spent years looking for the right way to multiply triples of real numbers:

$$(a, b, c) \times (d, e, f) = (??, ??, ??)$$

but was not able to solve the problem of the quotient of two triplets. In 1843, the answer dawned on him: he needed to use quadruples of numbers — quaternions — and also needed to be willing to abandon the commutative law of multiplication.

Very shortly after that, Cayley defined an 8-dimensional real division algebra, the octonions, which is the first and best-known example of a non-associative division algebra. The obvious question has to be how far this doubling process can be continued. In particular, can we describe a 16-dimensional real division algebra?

The answer is surprising: there isn't one. In 1878, Frobenius proved that the only associative real division algebras are the real numbers, the complex numbers, and the quaternions. Hurwitz in 1898 proved that the algebras of real numbers, complex numbers, quaternions, and Cayley numbers are the only examples where multiplication by unit vectors is distance-preserving.

A celebrated theorem of Bott, Milnor, and Kervaire from 1958 asserts that any "topological division algebra" (a vector space with a continuous multiplication that has no zero divisors), if it is finite dimensional over the reals, then it must have dimension 1, 2, 4 or 8. Kind of a shocking result, established using brand-new-at-the-time methods from algebraic topology.

This talk will be about all this and more.

Dave Boyles received the Ph.D. from the University of Wisconsin in 1986 and has taught at UWP since 1990. His research involves computations on algebraic curves, but he is also fascinated by the history of mathematics and the development of mathematical thinking. When not doing mathematics, he enjoys flower gardening, the Stanley Cup playoffs, progressive rock, spending as much time as possible with his grandson, and driving around Ireland, searching for the best view of the Atlantic Ocean and the perfect pint of Guinness.